Implicit differentiation

Implicit differentiation allows you to take the derivative of a function that contains both x and y on the same side of the equation. If you can't solve the function for y, implicit differentiation is the only way to take the derivative.

On the left sides of these derivatives, instead of seeing y' or $f'(x)$, you'll find dy/dx instead. In this notation, the numerator tells you what function you're deriving, and the denominator tells you what variable is being derived. dy/dx is literally read "the derivative of y with respect to x."

One of the most important things to remember, and the thing that usually confuses students the most, is that we have to treat y as a function and not just as a variable like x. Practically, this means that each time we take the derivative of a term that includes y, we multiply the result by dy/dx. To use implicit differentiation, follow these steps:

1. Differentiate both sides with respect to x.
2. Whenever you encounter y, differentiate it like you would x, but then multiply that term by dy/dx.
3. Move all terms involving dy/dx to the left side and everything else to the right.
4. Factor out dy/dx on the left and divide both sides by the other left-side factor so that dy/dx is the only thing remaining on the left.

Example

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Use implicit differentiation to find the derivative.

$$x^3 + y^3 = 9xy$$

Our first step is to differentiate both sides with respect to x. Treat y as a variable just like x, but whenever you take the derivative of a term that includes y, multiply by dy/dx. You'll need to use the product rule for the right side, treating $9x$ as one function and y as another.

$$3x^2 + 3y^2\frac{dy}{dx} = (9)(y) + (9x)(1)\frac{dy}{dx}$$

$$3x^2 + 3y^2\frac{dy}{dx} = 9y + 9x\frac{dy}{dx}$$

Move all terms that include dy/dx to the left side, and move everything else to the right side.

$$3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} = 9y - 3x^2$$

Factor out dy/dx on the left, then divide both sides by $\left(3y^2 - 9x\right)$.

$$\frac{dy}{dx}\left(3y^2 - 9x\right) = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x}$$

Dividing the right side by 3 to simplify gives us our final answer.

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$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}$$

Equation of the tangent line

We'll do another complete example in the next section, but let's get a preview of what it looks like to find the equation of the tangent line for an implicitly-defined function.

Just for fun, let's pretend you're asked to find the equation of the tangent line of the function in the previous example at the point $(2,4)$.

You'd just pick up right where you left off, and plug in this point to the derivative of the function to find the slope of the tangent line.

Example (continued)

$$\frac{dy}{dx}(2,4) = \frac{3(4) - (2)^2}{(4)^2 - 3(2)}$$

$$\frac{dy}{dx}(2,4) = \frac{12 - 4}{16 - 6}$$

$$\frac{dy}{dx}(2,4) = \frac{8}{10} = \frac{4}{5}$$

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Since you have the point $(2,4)$ and the slope of the tangent line at the point $(2,4)$, plug the point and the slope into point-slope form to find the equation of the tangent line. Then simplify.

$$y-4=\frac{4}{5}(x-2)$$

$$5y-20=4(x-2)$$

$$5y-20=4x-8$$

$$5y=4x+12$$

$$y=\frac{4}{5}x+\frac{12}{5}$$

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